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Reentrant enhancement of quantum fluctuations for symmetric environmental coupling

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The *system-plus-reservoir* (SPR) model is the most common and effective approach to study quantum dissipative effects. Indeed, it makes quantization possible by considering the whole energy-conserving system, while the reservoir's degrees of freedom, assumed to be harmonic, can be *traced out* by the path-integral technique, leading to a formulation that only includes the system of interest. In the standard SPR model the environment is only coupled with the system's coordinate and turns out to quench its quantum fluctuations. However, there are physical systems coupled with an environment whose "coordinates" and "momenta" can be completely interchangeable (e.g., magnets), so an SPR coupling must symmetrically affect both canonical variables. In this paper such a general environmental coupling is studied in the case of a harmonic oscillator. It is found that quantum fluctuations are generally enhanced by environmental coupling with an unexpected nonmonotonic behavior. This leads one to speculate about the possibility that spin-lattice coupling could drive the two-dimensional Heisenberg antiferromagnet to reach its quantum-critical point.

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I. INTRODUCTION

The most popular approach to the theory of quantum dissipative systems [1] involves the *system-plus-reservoir* (SPR) model [2–5], where dissipation is assumed to arise from the interaction with a *reservoir* (or environment or bath); the latter is constituted by very many degrees of freedom which are individually weakly perturbed by the system so that (a) the reservoir can be assumed to remain at thermal equilibrium and (b) its degrees of freedom can be modeled as harmonic oscillators, which are (c) linearly coupled with some system's observable. This observable is usually taken as a function of the system's coordinate only, and the case of *strictly linear* dissipation corresponds to the case of a linear function.

Within such a model it is possible to show that the (retarded) dynamics of the system, once the bath variables are eliminated, is indeed described by a Langevin equation [6], where the dissipative memory function is connected to the characteristics of the bath and its coupling to the system. This contact between the microscopic details of the bath and the phenomenological description is valuable as it allows us to connect with the phenomenology the effects onto the quantum thermodynamics of the system.

Originally, the picture of a more or less fictitious harmonic bath was meant to give a physically sound tool for describing quantum dissipation: a Hamiltonian quantization is made possible because the overall system including the bath is isolated. However, in several real cases it is possible to precisely devise the physical environment and its microscopic Hamiltonian. For instance, a magnetic system is intrinsically coupled with its underlying lattice; this is demonstrated by the observed occurrence of effects that can be as dramatic as to yield a lattice distortion (*spin-Peierls* transition [7,8]). Now, while the motion of the lattice ions can be fairly described in terms of linear excitations (phonons), it is also possible to microscopically model the coupling between the spins and the ion positions so that the vibrating lattice can be eventually treated as an environmental bath for the spins. Within this picture one could expect and theoretically calculate observable modifications of the behavior of quantum magnets, such as a shift of the predicted critical temperature [9–11]. As spin Hamiltonians are often dealt with through a spin-boson transformation reducing spin operators to ordinary canonical variables, thinking of a possible environmental coupling would symmetrically involve both the coordinates and the momenta, with no privileged role: this has been shown in Ref. [12] for the case of the easy-axis *XXZ* magnet. Therefore it is necessary to generalize the concept of the dissipative system in a much more general way than it was done when considering *anomalous* dissipation [13–15], where the bath is assumed to be coupled with the momentum.

Such a generalization has been used for studying the coherence properties of a magnetic impurity in a ferromagnetic environment [16]: it has been shown that some quantum effects of the two baths partially cancel, leading to a persistence of coherence that has been called *quantum frustration* of decoherence [17] and this motivated for a more detailed study of the dynamics of the quantum oscillator coupled with two baths [18].

The aim of the present paper is to provide a deeper understanding of the thermodynamic behavior that follows from the intrinsic dynamical character of quantum thermodynamics. Indeed, besides the dynamical features studied in Ref. [18], one expects that also the system's static quantities could exhibit effects that witness for *quantum frustration*. The present work gives an overview of the different—

sometimes unexpected—features that a generalized environmental coupling can produce.

From previous studies it is well known that, roughly speaking, the quantum fluctuations of the variable to which the bath is attached are quenched and those of its conjugate variable are enhanced, in such a way that the Heisenberg uncertainty principle holds. However, it is not clear which would be the prevailing effect when attaching one or two independent baths to both canonical variables, especially when the coupling has an equivalent weight onto coordinate and momentum (*symmetric coupling*). We are going here to answer such questions by assuming the simultaneous existence of environmental coupling with both canonical variables, taking an exactly solvable reference system, i.e., the harmonic oscillator. We will find that the presence of a symmetric environmental coupling generally enhances the quantum fluctuations of coordinate and momentum, but that this does not necessarily occur in a monotonic way with the coupling strength. Moreover, as such phenomena can show up in a qualitatively similar way in nonlinear systems, one can think of the possibility of finding reentrant behavior driven by the environmental coupling strength, e.g., when the system is close to a (quantum) phase transition.

In Sec. II the framework of general environmental coupling is introduced considering two independent reservoirs coupled to the coordinate and to the momentum, while in Sec. III the alternative case of one single bath coupled to both canonical variables is approached; there, the main results for the environmental effects upon the mean-square fluctuations of the harmonic oscillator are reported and discussed for coupling with the only environment coordinates and also when the environmental momenta are involved. In Sec. IV we summarize the results and draw some conclusions and speculations about the possible implications of symmetric environmental coupling. In Appendix A the backbone of the standard theory of quantum dissipation is briefly reviewed by recalling the definition of relevant quantities and setting the formalism adopted in this paper; Appendix B reports the details of the evaluation of the general Gaussian path integral repeatedly used throughout the paper.

II. COORDINATE AND MOMENTUM COUPLED WITH TWO INDEPENDENT ENVIRONMENTS

We consider as system of interest a quantum harmonic oscillator,

$$\hat{\mathcal{H}}_{\rm S} = \frac{1}{2}(a^2\hat{p}^2 + b^2\hat{q}^2), \tag{1}$$

where we prefer to introduce two parameters a and b in the place of the commonly used mass $m = a^{-2}$ and frequency $\omega_0 = ab$, in such a way to emphasize the symmetry between $\hat{p}$ and $\hat{q}$. In addition, we assume $\hbar = 1$, i.e., the commutator $[\hat{q},\hat{p}] = 1$.

A. Influence action and fluctuations

As motivated in the Introduction, we wish here consider the case of two baths, one coupled with the coordinate and one with the momentum. The known results summarized in Appendix A tell us that the two independent baths are in conflict, so the first natural but nontrivial question is which effect prevails onto the mean-square fluctuations of, say, the coordinate. Of course, the answer also depends on the relative intensities of the couplings; however, if the bath couplings are *identical*, the behavior is not easily foreseeable.

We start then by coupling the canonical variables of the oscillator Hamiltonian (1) with two independent baths, i.e.,

$$\hat{\mathcal{H}}_{\rm I} = \frac{1}{2}\sum_\alpha [a_\alpha^2\hat{p}_\alpha^2 + b_\alpha^2(\hat{q}_\alpha - \hat{q})^2] + \frac{1}{2}\sum_\beta [a_\beta^2(\hat{p}_\beta - \hat{p})^2 + b_\beta^2\hat{q}_\beta^2], \tag{2}$$

here the subscripts are used to distinguish between the two independent sets of bath parameters $\{a_\alpha, b_\alpha\}$ and $\{a_\beta, b_\beta\}$, corresponding to the frequencies $\{\omega_\alpha = a_\alpha b_\alpha\}$ and $\{\omega_\beta = a_\beta b_\beta\}$. In analogy with Eq. (A4) we are thus lead to introduce two "memory functions," $\gamma_p(t)$ and $\gamma_q(t)$, whose Laplace transforms read as

$$\gamma_p(z) = z\sum_\beta \frac{a_\beta^2}{z^2 + \omega_\beta^2}, \qquad \gamma_q(z) = z\sum_\alpha \frac{b_\alpha^2}{z^2 + \omega_\alpha^2}. \tag{3}$$

The equations of motion for the system's coordinate can be again put into the form of a Langevin equation,

$$\ddot{\hat{q}}(t) + \int dt'\,\Gamma(t')\dot{\hat{q}}(t - t') + \omega_0^2\hat{q}(t) = \hat{F}(t), \tag{4}$$

where

$$\Gamma(t) = b^2\gamma_p(t) + a^2\gamma_q(t) + \dot{\gamma}_{pq}(t) \tag{5}$$

and

$$\gamma_{pq}(t) = \int dt'\,\gamma_p(t')\gamma_q(t - t'). \tag{6}$$

Assuming $\gamma_q(t)$ and $\gamma_p(t)$ to vanish for $t < 0$ and to be positive decreasing functions for $t > 0$, then $\Gamma(t)$ also satisfies the condition of causality, but as $\dot{\gamma}_{pq}(t) < 0$ the positivity of $\Gamma(t)$ is not guaranteed. Basically, this means that there is not a standard phenomenological dissipative counterpart of our *system-plus-two-reservoirs* model. Therefore, it is better to refer to the two-bath coupling as *environmental*, rather than *dissipative*, coupling. Of course, in physical applications the baths and their interaction with the system have to be microscopically characterized. Anyhow, since in the limit where one of the baths can be disregarded we expect a (standard or anomalous) dissipative behavior, we argue that a phenomenological form for both memory functions is physically reasonable; we will make use of it later on.

The calculation of the influence action can be made separately for the two baths in the very same way that leads to Eq. (A2),

$$\begin{aligned}\mathcal{S}_{\mathrm{I}}[p,q] &= -\frac{1}{2}\int_0^\beta du\,du'[k_p(u-u')p(u)p(u') \\ &\quad + k_q(u-u')q(u)q(u')] \\ &= -\frac{\beta}{2}\sum_n (k_{pn}p_np_{-n} + k_{qn}q_nq_{-n}),\end{aligned} \tag{7}$$

where the Matsubara components of the kernels $k_p(u)$ and $k_q(u)$ read

$$k_{pn} = \sum_\beta \frac{a_\beta^2\nu_n^2}{\nu_n^2 + \omega_\beta^2}, \quad k_{qn} = \sum_\alpha \frac{b_\alpha^2\nu_n^2}{\nu_n^2 + \omega_\alpha^2} \tag{8}$$

so that they are connected with the memory functions $\gamma_p(z)$ and $\gamma_q(z)$ by a relationship analogous to Eq. (A5).

Eventually, including the isolated system's action and using the general result of Appendix B one finds the partition function

$$\mathcal{Z} = \frac{1}{\beta\omega_0}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\nu_n^2 + (a^2+k_{pn})(b^2+k_{qn})}, \tag{9}$$

and the mean-square fluctuations

$$\langle\hat{p}^2\rangle = \frac{1}{\beta}\sum_n \frac{b^2+k_{qn}}{\nu_n^2 + (a^2+k_{pn})(b^2+k_{qn})},$$

$$\langle\hat{q}^2\rangle = \frac{1}{\beta}\sum_n \frac{a^2+k_{pn}}{\nu_n^2 + (a^2+k_{pn})(b^2+k_{qn})}, \tag{10}$$

which, as expected, are symmetrically related and reduce to the known forms in the limits of standard ($k_{pn}\to 0$) and anomalous ($k_{qn}\to 0$) dissipation. The summations in Eqs. (10) are convergent, provided that for large n both kernels are such that $k_n/n\to 0$. Together with the result $\langle\hat{p}\hat{q}+\hat{q}\hat{p}\rangle = 0$, Eqs. (10) fully determine the Gaussian (reduced) density matrix corresponding to the oscillator Hamiltonian (1) plus the environmental interaction [Eq. (2)].

The zeroth Matsubara component of Eqs. (10) coincides with the classical contribution to the mean-square fluctuations

$$\langle\hat{p}^2\rangle_{\mathrm{cl}} = \frac{1}{\beta a^2} = \frac{m}{\beta}, \quad \langle\hat{q}^2\rangle_{\mathrm{cl}} = \frac{1}{\beta b^2} = \frac{1}{\beta m\omega^2}, \tag{11}$$

noting that the classical contribution is unaffected by the environment, it is useful to separate the remaining *purely quantum* contribution, which includes the whole environmental effect,

$$\langle\hat{p}^2\rangle_{\mathrm{pq}} = \frac{2}{\beta}\sum_{n=1}^{\infty} \frac{b^2+k_{qn}}{\nu_n^2 + (a^2+k_{pn})(b^2+k_{qn})},$$

$$\langle\hat{q}^2\rangle_{\mathrm{pq}} = \frac{2}{\beta}\sum_{n=1}^{\infty} \frac{a^2+k_{pn}}{\nu_n^2 + (a^2+k_{pn})(b^2+k_{qn})}. \tag{12}$$

For the isolated oscillator these quantities reduce to

$$\langle\hat{p}^2\rangle_{\mathrm{pq}} = \frac{b}{2a}\left(\coth\frac{\beta\omega_0}{2} - \frac{2}{\beta\omega_0}\right),$$

$$\langle\hat{q}^2\rangle_{\mathrm{pq}} = \frac{a}{2b}\left(\coth\frac{\beta\omega_0}{2} - \frac{2}{\beta\omega_0}\right), \tag{13}$$

which look more familiar by noting that $b/a = m\omega_0$.

B. Environmental effects

Let us consider the general expressions (12) for the purely quantum mean-square fluctuations. Due to the interchangeability of the canonical variables, it is sufficient to study, say, the coordinate fluctuations only.

To proceed, we take the minimal phenomenological form for the memory function of each bath, i.e., the *Drude* model [1], defined by $\gamma(t) = \gamma\tau_{\mathrm{D}}^{-1}e^{-t/\tau_{\mathrm{D}}}$ and characterized by the intensity γ and the memory time τ_{D}. Its Laplace transform, to which the bath's kernel is related via Eq. (A5), reads

$$\gamma(z) = \frac{\gamma}{1+\tau_{\mathrm{D}}z}. \tag{14}$$

The Drude cut-off frequency $\omega_{\mathrm{D}} = \tau_{\mathrm{D}}^{-1}$ characterizes the environment and its interaction with the system, so it is expected to be of the order of the Debye frequency in the case of a phonon bath. In the limit of vanishing memory time, i.e., large ω_{D}, one recovers the so-called *Ohmic* (or *Markovian*) model, which is memoryless, $\gamma(t) = \gamma\delta(t-0^+)$. Actually, a finite Drude frequency is necessary in order to get finite results for both quantities in Eqs. (12); on the other hand, it is indeed known that Ohmic response is incompatible with quantum mechanics [19,20].

The Drude kernels we are going to use are then

$$k_{pn} = \gamma_p\frac{|\nu_n|}{1+\tau_{\mathrm{D}}|\nu_n|}, \quad k_{qn} = \gamma_q\frac{|\nu_n|}{1+\tau_{\mathrm{D}}|\nu_n|}, \tag{15}$$

where for both kernels the same memory time τ_{D} is chosen for simplicity, while the amplitudes γ_p and γ_q will be varied independently. In addition, without loss of generality we can take a symmetric form for the oscillator Hamiltonian (1), i.e., $a^2 = b^2 = \omega_0$; in this way both γ_q and γ_p are dimensionless. Taking ω_0 as the frequency unit leads to a natural dimensionless formulation, with the reduced temperature $t = 1/(\beta\omega_0)$, the Matsubara frequency $\nu_n = 2\pi tn\omega_0$, and so on; all this is tantamount to setting $\omega_0 = 1$, and in such units the classical part of the dimensionless fluctuations is equal to the temperature, $\langle\hat{q}^2\rangle_{\mathrm{cl}} = t$.

Therefore, the mean-square fluctuation we have to study is

$$\langle\hat{q}^2\rangle_{\mathrm{pq}} \equiv \alpha(t,\gamma_p,\gamma_q,\tau_{\mathrm{D}}), \tag{16}$$

which has the dimensionless explicit expression

$$\alpha = 2t\sum_{n=1}^{\infty}\frac{1+k_{pn}}{(2\pi tn)^2 + (1+k_{pn})(1+k_{qn})}, \tag{17}$$

with

$$k_{pn} = \gamma_p\frac{2\pi tn}{1+\tau_{\mathrm{D}}2\pi tn}, \quad k_{qn} = \gamma_q\frac{2\pi tn}{1+\tau_{\mathrm{D}}2\pi tn}. \tag{18}$$

Note that a nonzero memory time τ_{D} is required to make the series [Eq. (17)] convergent.

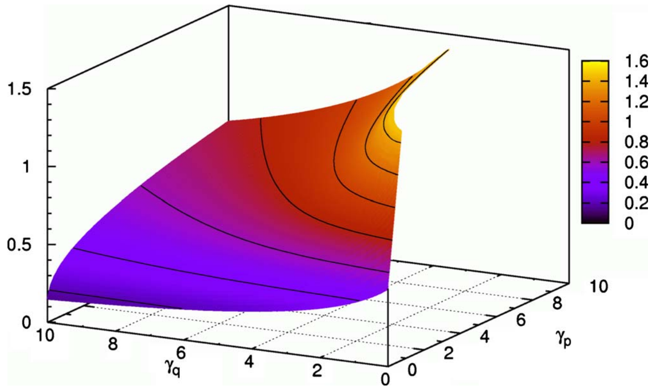

FIG. 1. (Color online) Mean-square fluctuation of the coordinate at $t=0$ as a function of both γ_q and γ_p, for $\tau_\mathrm{D}=0.01$.

1. Zero temperature

Taking the limit $t \rightarrow 0$, one sets $x=2\pi tn$ and $dx=2\pi t$ to write Eq. (17) as an integral,

$$\alpha(0,\gamma_p,\gamma_q,\tau_\mathrm{D}) = \int_0^\infty \frac{dx}{\pi} \frac{1+\gamma_p g(x)}{x^2+[1+\gamma_p g(x)][1+\gamma_q g(x)]}, \tag{19}$$

where

$$g(x) = \frac{x}{1+\tau_\mathrm{D} x}. \tag{20}$$

From Eq. (19) one can see that the result of the isolated-system agrees with Eq. (13) in dimensionless form,

$$\alpha(0,0,0,\tau_\mathrm{D}) = \frac{1}{\pi}\int \frac{dx}{x^2+1} = \frac{1}{2}, \tag{21}$$

and also recover a known analytic result [1] in the case of Ohmic standard dissipation,

$$\begin{aligned}\alpha(0,0,\gamma,0) &= \frac{2}{\pi\sqrt{4-\gamma^2}}\tan^{-1}\frac{\sqrt{4-\gamma^2}}{\gamma}(\gamma<2)\\ &= \frac{2}{\pi\sqrt{\gamma^2-4}}\tanh^{-1}\frac{\sqrt{\gamma^2-4}}{\gamma}(\gamma>2).\end{aligned} \tag{22}$$

In the general case Eq. (19) has to be evaluated numerically. This is what we have done and Fig. 1 shows the three-dimensional resulting plot for $\alpha(0,\gamma_p,\gamma_q,\tau_\mathrm{D}=0.01)$; its shape gives an overall idea of the combined effect of two baths. However, in order to highlight some important features which are barely apparent, it is necessary to analyze its cross sections.

In Fig. 2 we first consider how α varies while increasing the coordinate coupling γ_q: also for nonzero γ_p the effect is a decrease in α, as if the q bath were a position measuring device, causing localization of the particle. From the same figure, and more clearly from Fig. 3, the complementary interpretation can be given to the action of the p bath, generalizing the observations made in the case of *anomalous* dissipation [14,15]; note, however, that as soon as γ_q is switched on the steepness of the rise of α is considerably suppressed.

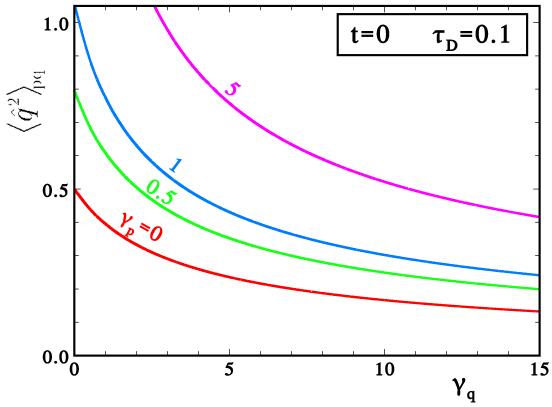

FIG. 2. (Color online) Mean-square fluctuation of the coordinate at $t=0$ as a function of γ_q for fixed values of γ_p. As in the case of standard dissipation, $\gamma_p=0$, the effect of increasing the intensity of the coordinate coupling is a decrease in $\langle\hat{q}^2\rangle$.

Given the opposite effect of the two baths, it is natural to ask what the result will be when both have *equal* strength, $\gamma_p=\gamma_q$: this is a nontrivial issue, especially from the point of view of the application to real problems where the canonical variables play symmetric roles, as mentioned in the Introduction.

The answer is given by the curves reported in Fig. 4: $\alpha=\langle\hat{q}^2\rangle_{pq}$ is larger compared to the isolated oscillator, i.e., the quantum fluctuations of both canonical variables are *enhanced* by a symmetric environmental coupling. However, an unexpected behavior shows up while rising the intensity of the environmental coupling: after an initial rise the mean-square fluctuation shows a maximum, whose location and intensity depends upon the Drude memory time, and eventually slowly decreases back toward the isolated oscillator value $\alpha=1/2$.

It is obvious that, as the size of the zero-temperature fluctuations is affected by the mechanism of environmental coupling, one could formally modulate its intensity γ to drive

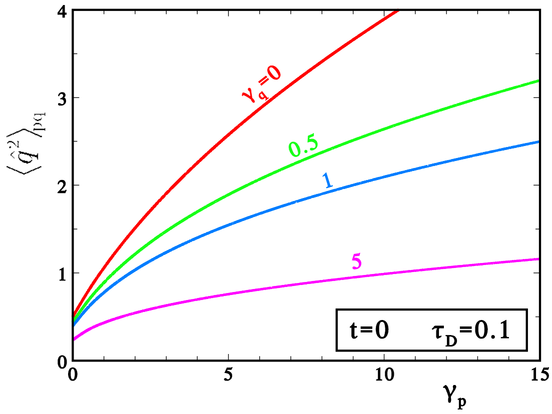

FIG. 3. (Color online) Mean-square fluctuation of the coordinate at $t=0$ as a function of γ_p for fixed values of γ_q. As in the case of anomalous dissipation, $\gamma_q=0$, the effect of increasing the intensity of the momentum coupling is an increase in $\langle\hat{q}^2\rangle$.

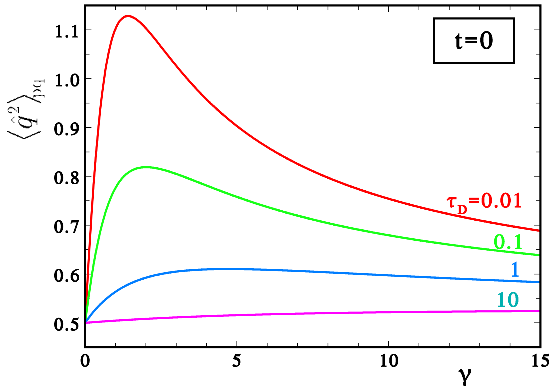

FIG. 4. (Color online) Mean-square fluctuation of the coordinate at $t=0$ as a function of $\gamma=\gamma_q=\gamma_p$, i.e., for equally acting environments, for selected values of the Drude time τ_{D}. It appears that between the competing effects of the two baths, the prevailing one is that of increasing $\langle\hat{q}^2\rangle$; however, as the coupling intensity increases a maximum is reached and followed by a slow decrease toward the isolated-oscillator value.

through a quantum phase transition (QPT) a system that is close to it. At difference with standard dissipative systems, e.g., resistively shunted Josephson junction arrays, where a stronger dissipation leads toward the ordered (superconducting) phase [21], in the generalized case a higher γ would most commonly increase disorder. One can also speculate about the fact that the nonmonotonicity displayed in Fig. 4 opens the possibility to observe reentrant behavior around a QPT, namely, by further rising γ beyond the critical value one could observe the restoration of the isolated-system's ordered phase.

Having ascertained that the combined effect of the two baths is opposite to that of the single q bath, it is natural to look for the evolution one observes by gradually switching the p bath coupling on. In Figs. 5 the behavior of α is followed along straight lines in the plane (γ_q,γ_p), i.e., for fixed ratios $c=\gamma_p/\gamma_q$ between the intensities of the two environmental couplings. These figures, which differ for the choice of the Drude time, are essentially similar and display the competing interplay between the intensities of the environmental couplings for q and p: if the latter is small enough, the quenching effect of the former prevails. Increasing γ_p for any fixed γ_q, α must cross the isolated-system value $\alpha=1/2$ for some $\tilde{\gamma}_p$: along the line $\tilde{\gamma}_p(\gamma_q)$ the competing effects cancel each other and the mean-square fluctuation of the coordinate is stationary; since $\alpha>1/2$ for $\gamma_p=\gamma_q$ it follows that $\tilde{\gamma}_p(\gamma_q)<\gamma_q$. From Eq. (19) one can analytically obtain the initial slope of these curves as a function of the Drude memory time τ_{D},

$$\tilde{\gamma}_p'(0)=\frac{2-\pi\tau_{\mathrm{D}}-4\tau_{\mathrm{D}}^2\ln\tau_{\mathrm{D}}+2\tau_{\mathrm{D}}^2+\pi\tau_{\mathrm{D}}^3}{-4\ln\tau_{\mathrm{D}}-2+3\pi\tau_{\mathrm{D}}-2\tau_{\mathrm{D}}^2+\pi\tau_{\mathrm{D}}^3},\qquad(23)$$

which turns out to vanish in the Ohmic limit and tends to 1 for large τ_{D}. This is in agreement with the numerical calculation of $\tilde{\gamma}_p(\gamma_q)$ reported in Fig. 6.

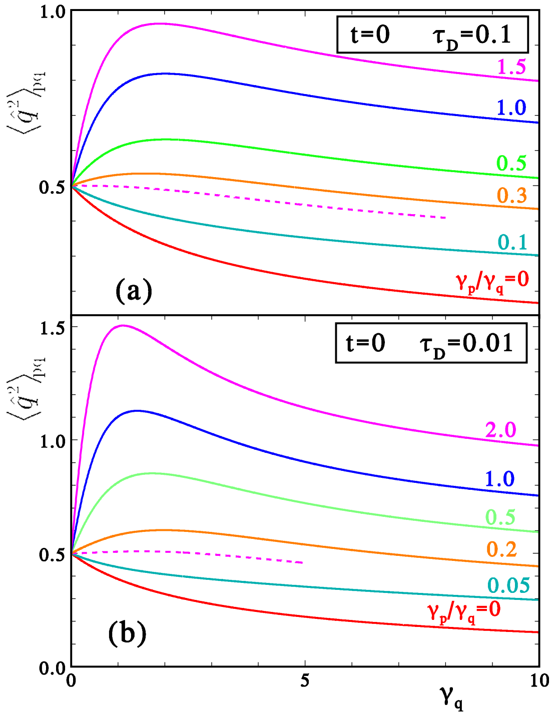

FIG. 5. (Color online) Mean-square fluctuation of the coordinate for $\tau_{\mathrm{D}}=0.1$ (a) and $\tau_{\mathrm{D}}=0.01$ (b) at $t=0$ along the line $\gamma_p=c\gamma_q$ as a function of γ_q, for different values of c. The dashed line for $c=0.2214$ in (a) and $c=0.1193$ in (b) separates the decreasing from the nonmonotonic behaviors.

Figure 5 also suggests that, keeping the ratio $c=\gamma_p/\gamma_q$ fixed, for large coupling intensity $\langle\hat{q}^2\rangle$ tends to a finite value; one can indeed find that

$$\langle\hat{q}^2\rangle\rightarrow\frac{1}{2}\sqrt{c}=\frac{1}{2}\sqrt{\frac{\gamma_p}{\gamma_q}}.\qquad(24)$$

Substantially, it turns out that, after a possible initial increase in the spread of one of the canonical variables when the

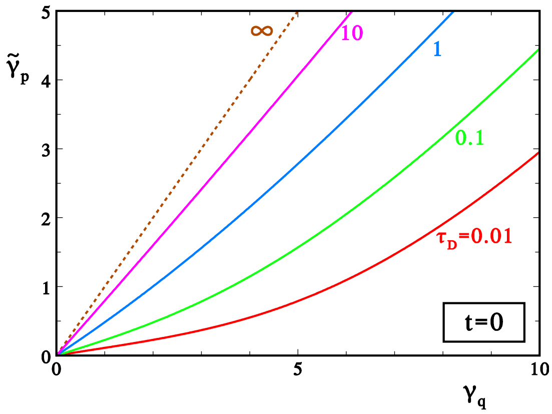

FIG. 6. (Color online) The function $\tilde{\gamma}_p(\gamma_q)$ corresponding to the cancellation of the competing effects of the two baths so that the mean-square fluctuation of the coordinate keeps the isolated system's value. The curves refer to $t=0$, so $\langle\hat{q}^2\rangle=1/2$, and to selected values of the Drude time τ_{D}.

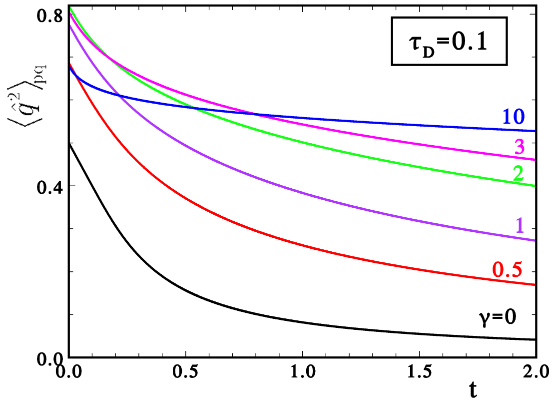

FIG. 7. (Color online) Pure-quantum mean-square fluctuation of the coordinate for $\tau_{\mathrm{D}}=0.1$ and different values of $\gamma=\gamma_q=\gamma_p$ (i.e., for equally acting environments) as a function of t.

interaction is switched on, a very strong environmental coupling eventually tends to localize again both $\hat{p}$ and $\hat{q}$, as the above limit also entails that

$$\langle\hat{p}^2\rangle \to \frac{1}{2}\sqrt{\frac{\gamma_q}{\gamma_p}}, \tag{25}$$

the shrinking of the fluctuations is visibly more effective for the variable with larger bath-coupling strength, but the product of the fluctuations obeys the constraint of the uncertainty principle,

$$\langle\hat{q}^2\rangle\langle\hat{p}^2\rangle \to \frac{1}{4}. \tag{26}$$

In other words, a very large environmental coupling leads to a situation where the uncertainty is *squeezed* along one axis in the p-q plane with respect to the symmetric spread in the free system.

2. Finite temperature

As Eq. (17) shows, the effect of environmental coupling only affects the purely quantum part of the fluctuations, which decreases with increasing temperature: so, one expects a softening of the effects observed at zero-T when T rises. However, the fate of the reentrant fluctuation amplitude displayed in Figs. 4 and 5 is not obvious. To unveil such issue we plot in Fig. 7 the behavior of $\alpha=\langle\hat{q}^2\rangle_{\mathrm{pq}}$ vs temperature for the situation of equally acting environments. It appears that a stronger environmental coupling weakens the decrease in $\langle\hat{q}^2\rangle_{\mathrm{pq}}$ with temperature and that for $t\gtrsim 1$ a monotonic behavior is apparently restored. This fact is more evident by looking at the curves for increasing temperatures reported in Fig. 8.

When the temperature is larger than the Drude cut-off frequency, $t \gg 1/(2\pi\tau_{\mathrm{D}})$, expressions (18) for the kernels can be well approximated by their asymptotic values $k_{qn} \simeq \gamma_q/\tau_{\mathrm{D}}$ and $k_{pn} \simeq \gamma_p/\tau_{\mathrm{D}}$, so their dependence on n disappears and summation (17) becomes analytic,

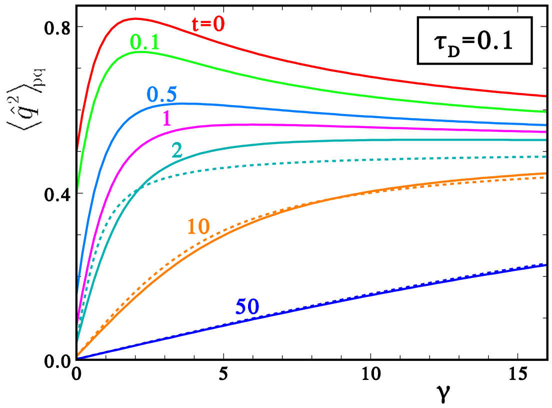

FIG. 8. (Color online) Pure-quantum mean-square fluctuation of the coordinate at different temperatures as a function of $\gamma=\gamma_q=\gamma_p$, for $\tau_{\mathrm{D}}=0.1$. The dashed curves for $t\gtrsim 2$ represent the large-t analytic approximation (27) valid for $t \gg 1/(2\pi\tau_{\mathrm{D}}) \simeq 1.6$.

$$\alpha \simeq \frac{c_p}{2c_q}\left(\coth\frac{c_p c_q}{2t} - \frac{2t}{c_p c_q}\right), \tag{27}$$

with $c_{p,q}^2 = 1+\gamma_{p,q}/\tau_{\mathrm{D}}$: this approximation is shown as dashed lines in Fig. 8; for $c_p c_q/2t \gg 1$, i.e., under the combined condition $\sqrt{\gamma_p\gamma_q} \gg 2\tau_{\mathrm{D}} t \gg 1/\pi$, the value of α tends to the constant $c_p/2c_q$. As the validity of this approximation entails the disappearance of the maximum, the nonmonotonic behavior is expected for $t \lesssim 1/(2\pi\tau_{\mathrm{D}})$.

III. COORDINATE AND MOMENTUM COUPLED WITH A SINGLE ENVIRONMENT

It can also happen that there is one single environment coupled with both canonical variables: the most general influence Hamiltonian can be found in Ref. [18]. However, there are two main scenarios of physical relevance: the first occurs when both $\hat{q}$ and $\hat{p}$ are coupled with the same bath variables (say, the coordinates), while in the second the q_α's are coupled with $\hat{q}$ and the p_α's with $\hat{p}$. The first scenario has a realization in the coupling of spins with lattice vibrations due to their modulation of the exchange integral; in the second case a natural model could be a spin-orbit interaction where the orbital angular momenta are thermalized by lattice vibrations.

A. System coupled with bath coordinates

Starting from the coupling Hamiltonian,

$$\hat{\mathcal{H}}_{\mathrm{I}} = \frac{1}{2}\sum_\alpha \left[a_\alpha^2\hat{p}_\alpha^2 + b_\alpha^2(\hat{q}_\alpha - c_\alpha\hat{q} - d_\alpha\hat{p})^2\right], \tag{28}$$

we get the influence action:

$$\mathcal{S}_{\mathrm{I}}[p,q] = -\frac{\beta}{2}\sum_n \left[k_{qn}q_n q_{-n} + k_{pn}p_n p_{-n} + 2\kappa_n p_n q_{-n}\right], \tag{29}$$

with the kernels

$$k_{qn} = \sum_\alpha \frac{b_\alpha^2 c_\alpha^2 \nu_n^2}{\nu_n^2 + \omega_\alpha^2}, \quad k_{pn} = \sum_\alpha \frac{b_\alpha^2 d_\alpha^2 \nu_n^2}{\nu_n^2 + \omega_\alpha^2}, \tag{30}$$

$$\kappa_n = \sum_\alpha \frac{b_\alpha^2 c_\alpha d_\alpha \nu_n^2}{\nu_n^2 + \omega_\alpha^2}. \tag{31}$$

It is immediate to check that the last kernel vanishes if $c_\alpha d_\alpha = 0$ for any α, yielding the previous Eq. (7): indeed, if the coefficients c_α and d_α are not simultaneously nonzero, the bath can be split into two independent baths and the two-bath case studied in the previous sections is recovered.

In the case when all the coefficients c_α and d_α are positive the spectral density [1] $J_{pq}(\omega)$ for κ_n, such that

$$\kappa_n = 2\nu_n^2 \int_0^\infty d\omega \frac{J_{pq}(\omega)}{\omega^2 + \nu_n^2} \tag{32}$$

can be expressed in terms of those for the first two kernels,

$$J_q(\omega) \equiv \sum_\alpha b_\alpha^2 c_\alpha^2 \omega_\alpha \delta(\omega^2 - \omega_\alpha^2),$$

$$J_p(\omega) \equiv \sum_\alpha b_\alpha^2 d_\alpha^2 \omega_\alpha \delta(\omega^2 - \omega_\alpha^2), \tag{33}$$

as their geometric average as their geometric average [18]

$$J_{pq}(\omega) = \sqrt{J_q(\omega) J_p(\omega)}. \tag{34}$$

The baths are independent if the supports of $J_q(\omega)$ and $J_p(\omega)$ have no intersection, i.e., if $J_{pq}(\omega)$ vanishes for any ω [22].

Eventually, using the result of Appendix B, the expressions for the fluctuations in the case of the bath coupling [Eq. (28)] are found,

$$\mathcal{Z} = \frac{1}{\beta\omega_0} \prod_{n=1}^\infty \frac{\nu_n^2}{\nu_n^2 + (a^2 + k_{pn})(b^2 + k_{qn}) - \kappa_n^2}, \tag{35}$$

$$\langle \hat{p}^2 \rangle_{\mathrm{pq}} = \frac{2}{\beta} \sum_{n=1}^\infty \frac{b^2 + k_{qn}}{\nu_n^2 + (a^2 + k_{pn})(b^2 + k_{qn}) - \kappa_n^2},$$

$$\langle \hat{q}^2 \rangle_{\mathrm{pq}} = \frac{2}{\beta} \sum_{n=1}^\infty \frac{a^2 + k_{pn}}{\nu_n^2 + (a^2 + k_{pn})(b^2 + k_{qn}) - \kappa_n^2},$$

$$\langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle_{\mathrm{pq}} = -\frac{4}{\beta} \sum_{n=1}^\infty \frac{\kappa_n}{\nu_n^2 + (a^2 + k_{pn})(b^2 + k_{qn}) - \kappa_n^2}. \tag{36}$$

As Eqs. (30) and (31) imply the inequality $k_{qn} k_{pn} \geq \kappa_n^2$, it follows that the denominators are always positive. Note that if the bath is coupled to the variable $\hat{p} \pm \hat{q}$ (i.e., $c_\alpha = \pm d_\alpha$), then $\pm\kappa_n = k_{pn} = k_{qn} > 0$: from the above variances it follows then that $\hat{p} \pm \hat{q}$ becomes smaller (larger) when the environmental coupling is switched on,

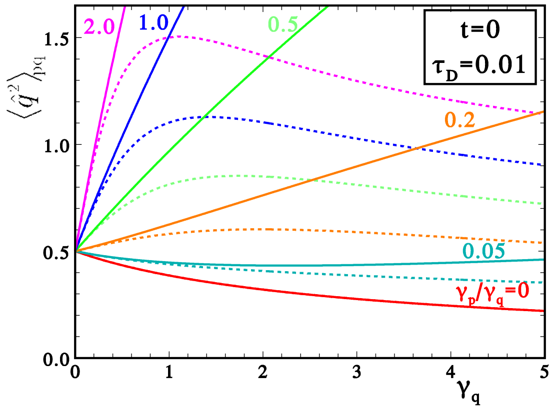

FIG. 9. (Color online) System coupled with the bath coordinates. Mean-square fluctuation of the coordinate, Eq. (40), at $t=0$ along the line $\gamma_p = c\gamma_q$ as a function of γ_q, for different values of c. The dashed lines correspond to the curves for the two-bath case of Fig. 5(b).

$$\langle (\hat{p} \pm \hat{q})^2 \rangle_{\mathrm{pq}} = \frac{2}{\beta} \sum_{n=1}^\infty \frac{a^2 + b^2 + (k_{pn} + k_{qn} \mp 2\kappa_n)}{\nu_n^2 + (a^2 + k_{pn})(b^2 + k_{qn}) - \kappa_n^2}, \tag{37}$$

which corresponds to a squeezed Gaussian distribution along a diagonal in the p-q plane, in agreement with the Heisenberg uncertainty relation. This observation also confirms the overall picture that the environment acts as a measuring device for whatever variable it is coupled with.

Let us consider now a phenomenological guess for the first two kernels [Eq. (30)], while the third one, κ_n, follows from Eqs. (32)–(34). Taking Drude kernels with equal memory times τ_{D} as in Eq. (15), corresponding to the spectral densities

$$J_{p,q}(\omega) = \frac{\gamma_{p,q}}{\pi} \frac{1}{1 + \tau_{\mathrm{D}}^2 \omega^2}, \tag{38}$$

one simply finds

$$\kappa_n = \frac{\sqrt{\gamma_q \gamma_p} |\nu_n|}{1 + \tau_{\mathrm{D}} |\nu_n|}, \tag{39}$$

in this case one exactly has $\kappa_n^2 = k_{pn} k_{qn}$, and in the dimensionless formulation introduced in Sec. II B the coordinate pure-quantum fluctuation is given by

$$\alpha = 2t \sum_{n=1}^\infty \frac{1 + k_{pn}}{(2\pi t n)^2 + 1 + k_{pn} + k_{qn}}, \tag{40}$$

with the kernels given in Eq. (18). For $t=0$ it becomes the integral

$$\alpha(0, \gamma_p, \gamma_q, \tau_{\mathrm{D}}) = \int_0^\infty \frac{dx}{\pi} \frac{1 + \gamma_p g(x)}{x^2 + 1 + (\gamma_p + \gamma_q) g(x)}, \tag{41}$$

with $g(x)$ as in Eq. (20).

In Fig. 9 it appears indeed that the mean-square fluctuation of the coordinate is enhanced by the coupling mechanism considered in this case, a result of the diagonal squeezing discussed after Eq. (37). Of course, different distributions of the coefficients c_α and d_α can yield a less dramatic increase with the environmental coupling strength.

B. System coupled with bath coordinates and momenta

In this case the most general interaction Hamiltonian would take the form

$$\hat{\mathcal{H}}_{\rm I}=\frac{1}{2}\sum_\alpha \left[a_\alpha^2(\hat{p}_\alpha - c_\alpha \hat{p})^2 + b_\alpha^2(\hat{q}_\alpha - c_\alpha \hat{q})^2\right], \tag{42}$$

note that in this case the coefficients c_α cannot be absorbed by means of linear canonical transformations of the bath variables.

The calculation of the influence action leads to an expression similar to Eq. (7), but for the appearance of an antisymmetric correlation kernel $\nu_n\tilde{\kappa}_n$,

$$\mathcal{S}_{\rm I}[p,q] = -\frac{\beta}{2}\sum_n \left(k_{pn}p_n p_{-n}+k_{qn}q_n q_{-n}+2\nu_n\tilde{\kappa}_n p_n q_{-n}\right), \tag{43}$$

with the Matsubara components

$$k_{qn}=\sum_\alpha \frac{c_\alpha^2 b_\alpha^2 \nu_n^2}{\nu_n^2+\omega_\alpha^2}, \qquad k_{pn}=\sum_\alpha \frac{c_\alpha^2 a_\alpha^2 \nu_n^2}{\nu_n^2+\omega_\alpha^2}, \tag{44}$$

$$\tilde{\kappa}_n = \sum_\alpha \frac{c_\alpha^2 \omega_\alpha^2}{\nu_n^2+\omega_\alpha^2}. \tag{45}$$

Using the result of Appendix B one has the expressions for the partition function and the mean-square fluctuations:

$$\mathcal{Z}=\frac{1}{\beta\omega_0}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\nu_n^2(1+\tilde{\kappa}_n)^2+(a^2+k_{pn})(b^2+k_{qn})}, \tag{46}$$

$$\langle \hat{p}^2\rangle_{\rm pq}=\frac{2}{\beta}\sum_{n=1}^{\infty}\frac{b^2+k_{qn}}{\nu_n^2(1+\tilde{\kappa}_n)^2+(a^2+k_{pn})(b^2+k_{qn})},$$

$$\langle \hat{q}^2\rangle_{\rm pq}=\frac{2}{\beta}\sum_{n=1}^{\infty}\frac{a^2+k_{pn}}{\nu_n^2(1+\tilde{\kappa}_n)^2+(a^2+k_{pn})(b^2+k_{qn})}, \tag{47}$$

while $\langle \hat{p}\hat{q}+\hat{q}\hat{p}\rangle=0$. Therefore, the kernel $\tilde{\kappa}_n$ has the effect to lower the fluctuations of both canonical variables with respect to the case of two independent baths. It is reasonable to use a phenomenological guess for k_{qn} and k_{pn}, while $\tilde{\kappa}_n$ is connected with them: in terms of the spectral densities it can be represented as

$$\tilde{\kappa}_n = 2\int_0^\infty \frac{\omega\sqrt{J_q(\omega)J_p(\omega)}}{\omega^2+\nu_n^2}\,d\omega. \tag{48}$$

Taking Drude kernels with equal memory times $\tau_{\rm D}$ as in Eq. (15), corresponding to the spectral densities [Eq. (38)], one obtains

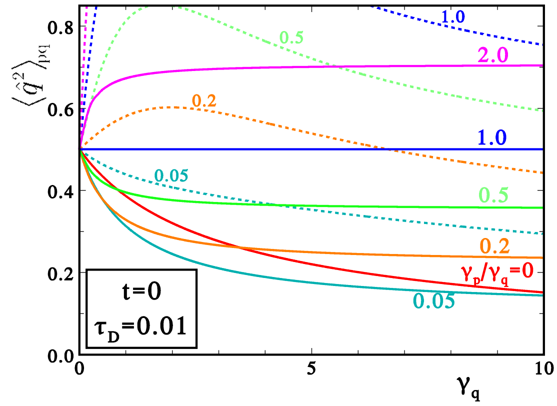

FIG. 10. (Color online) System coupled with the bath coordinates and momenta. Mean-square fluctuation of the coordinate [Eq. (50)] at $t=0$ along the line $\gamma_p = c\gamma_q$ as a function of γ_q, for different values of c. The dashed lines correspond to the curves for the two-bath case of Fig. 5(b).

$$\tilde{\kappa}_n = \sqrt{\gamma_q\gamma_p}\,f(\tau_{\rm D}\nu_n), \qquad f(x)\equiv \frac{1}{\pi}\frac{\ln x^2}{x^2-1}. \tag{49}$$

Note that $\tilde{\kappa}_n$ is positive and that $\nu_n\tilde{\kappa}_n \sim (\ln n)/n \to 0$ for $n\to\infty$. In the dimensionless formulation of Sec. II B, with the kernels [Eq. (18)], the pure-quantum fluctuation of the coordinate reads as

$$\alpha = 2t\sum_{n=1}^{\infty}\frac{1+k_{pn}}{(2\pi t n)^2(1+\tilde{\kappa}_n)^2+(1+k_{pn})(1+k_{qn})}, \tag{50}$$

with $\tilde{\kappa}_n = \sqrt{\gamma_q\gamma_p}\,f(2\pi t n\tau_{\rm D})$. At zero temperature this becomes the integral

$$\alpha(0,\gamma_p,\gamma_q,\tau_{\rm D}) = \int_0^\infty \frac{dx}{\pi}\frac{1+\gamma_p g(x)}{x^2[1+\tilde{\kappa}(x)]^2+[1+\gamma_p g(x)][1+\gamma_q g(x)]}, \tag{51}$$

with $\tilde{\kappa}(x)=\sqrt{\gamma_q\gamma_p}\,f(\tau_{\rm D}x)$ and $g(x)$ as in Eq. (20)

From Fig. 10 one can see that an environmental coupling of the kind of Eq. (42) produces much smaller effects than the previously considered ones. Note that the effect of increasing the bath-momentum coupling γ_p even initially shrinks the coordinate fluctuations, with a further slow rise. Moreover, it is particularly surprising to see that the existence of environmental coupling is irrelevant in the symmetric case; this means that the integral [Eq. (51)] sticks to the isolated system's value 1/2, whatever the values of $\gamma_p=\gamma_q$ and $\tau_{\rm D}$: this result is far from being apparent from Eq. (51).

IV. CONCLUSIONS

Motivated by quest for a suitable description of spin-lattice coupling, with the lattice regarded as a harmonic bath of thermalized degrees of freedom, we considered a generalized *system-plus-reservoir* (SPR) model where both coordinates and momenta are subject to the interaction with the environment. While it was known that coupling a bath to a canonical variable quenches its fluctuations, attaching independent baths to both coordinate and momentum generally enhances their fluctuations. In the case of a spin system, such larger quantum fluctuations could open the possibility of reaching the quantum phase transition of the spin-$\frac{1}{2}$ two-dimensional Heisenberg antiferromagnet [23], i.e., spin-lattice coupling could yield a vanishing zero-T sublattice magnetization with a power-law divergent correlation length. This possibility is to be further analyzed by a quantitatively accurate approach. Moreover and surprisingly, it turns out that the effect is nonmonotonic with the coupling strength, opening the possibility of a reentrant behavior in such a quantum critical system driven by its environmental coupling strength.

Other kinds of coupling involving both canonical variables and a single bath have been considered and briefly discussed. If the coupling terms do not simultaneously involve noncommuting bath variables (e.g., only bath coordinates) the result is a modification of the two-bath scenario with a further enhancement of the fluctuations and disappearance of the above mentioned nonmonotonicity. In the opposite case it clearly appears that the further "interference" effect gives rise to generally smaller fluctuations, i.e., the intensity of bath-momentum coupling is much less effective in increasing the coordinate fluctuations, which moreover show two exotic features: a counterintuitive initial decrease when γ_p is switched on and the intriguing stability to the isolated system's value in the symmetric case $\gamma_p=\gamma_q$; in other words, a symmetric environmental interaction appears to be ineffective for what attains mean-square fluctuations.

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We thank Professor Valerio Tognetti for stimulating us to further investigate these topics and for his useful suggestions.

APPENDIX A: THE STANDARD SYSTEM-PLUS-RESERVOIR MODEL

A bath attached to the coordinate $\hat{q}$ of the system of interest, e.g., the oscillator [Eq. (1)], is a collection of harmonic oscillators described by the Hamiltonian

$$\hat{\mathcal{H}}_{\mathrm{I}} = \frac{1}{2}\sum_{\alpha}\,[a_\alpha^2\hat{p}_\alpha^2 + b_\alpha^2(\hat{q}_\alpha - \hat{q})^2]. \tag{A1}$$

Its characteristics are "microscopically" specified by the collection of positive parameters $\{a_\alpha, b_\alpha\}$, the corresponding frequencies being $\omega_\alpha = a_\alpha b_\alpha$. At variance with the common use [1] we do not include here a third set of parameters c_α in front of $\hat{q}$, because they can be absorbed by means of a canonical transformation which affects the bath variables only, $\hat{q}_\alpha \to c_\alpha \hat{q}_\alpha$, $\hat{p}_\alpha \to c_\alpha^{-1}\hat{p}_\alpha$.

For the quantum thermodynamics at the equilibrium temperature $T=1/\beta$, the *influence action* $\mathcal{S}_{\mathrm{I}}[q]$ corresponding to Hamiltonian (A1) is obtained by *tracing out* the bath variables, i.e., by integrating over them in the corresponding path integral. The influence action is bilocal in imaginary time (a local contribution would arise from a Hamiltonian operator for the system and could not describe dissipation), but it is local (or diagonal) in Matsubara space [1]:

$$\mathcal{S}_{\mathrm{I}}[q] = -\frac{1}{2}\int_0^\beta du\,du'\,k(u-u')q(u)q(u')$$

$$= -\frac{\beta}{2}\sum_n k_n q_n q_{-n}, \tag{A2}$$

with the components q_n for each Matsubara frequency $\nu_n \equiv 2\pi n/\beta$ defined by $q(u) \equiv \Sigma_n q_n e^{-i\nu_n u}$. The Matsubara components of the *kernel* $k(u)$ are related to the bath parameters through

$$k_n \equiv \int_0^\beta du\,e^{i\nu_n u}k(u) = \nu_n^2\sum_\alpha \frac{b_\alpha^2}{\nu_n^2+\omega_\alpha^2}. \tag{A3}$$

Starting from the Hamiltonian $\hat{\mathcal{H}}=\hat{\mathcal{H}}_{\mathrm{S}}+\hat{\mathcal{H}}_{\mathrm{I}}$, after eliminating the bath variables from the equations of motion one finds that the dynamics is ruled by a quantum Langevin equation [6], where dissipation is described in terms of a (retarded) *memory function* $\gamma(t)$ whose Laplace transform is expressed in terms of the microscopic bath parameters as

$$\gamma(z) = \int_0^\infty dt\,e^{-zt}\gamma(t) = z\sum_\alpha \frac{b_\alpha^2}{z^2+\omega_\alpha^2}, \tag{A4}$$

hence, it is apparent that the kernel is related to the memory function,

$$k_n = |\nu_n|\,\gamma(z=|\nu_n|). \tag{A5}$$

This equality is a valuable relation as it connects the quantum dissipative kernel with a phenomenological quantity that can be experimentally accessed in a dissipative dynamical process.

The full system's action, i.e., the sum of the action $\mathcal{S}_{\mathrm{S}}$ of the isolated oscillator and the influence action $\mathcal{S}_{\mathrm{I}}$, is

$$\mathcal{S} = -\frac{\beta}{2}\sum_n\,[2\nu_n p_n q_{-n} + a^2 p_n p_{-n} + (b^2+k_n)q_n q_{-n}], \tag{A6}$$

and using the general result of Appendix B one finds the known results [1] for the partition function,

$$\mathcal{Z} = \frac{1}{\beta\omega_0}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\nu_n^2+a^2(b^2+k_n)}, \tag{A7}$$

and for the mean-square fluctuations of momentum and coordinate,

$$\langle\hat{p}^2\rangle = \frac{1}{\beta}\sum_n \frac{b^2+k_n}{\nu_n^2+a^2(b^2+k_n)},$$

$$\langle\hat{q}^2\rangle = \frac{1}{\beta}\sum_n \frac{a^2}{\nu_n^2+a^2(b^2+k_n)}. \tag{A8}$$

The above known results [1] qualitatively mean that standard dissipation decreases the quantum fluctuations of the coordinate, while it increases those of the momentum, compared to the limit of no dissipation ($k_n \to 0$), where the usual quantum expressions are recovered, i.e., $\mathcal{Z}=[2\sinh(\beta\omega_0/2)]^{-1}$ and $\langle\hat{p}^2\rangle/b^2=\langle\hat{q}^2\rangle/a^2=(1/2\omega_0)\coth(\beta\omega_0/2)$. Note that in the *Ohmic* case, where $k_n \sim n$ for large n, the fluctuation of $\hat{p}$ diverges, while that of $\hat{q}$ remains finite.

Of course, the symmetry of interchanging the canonical variables tells us that when the bath is coupled with the momentum, i.e., if

$$\hat{\mathcal{H}}_{\mathrm{I}}=\frac{1}{2}\sum_{\alpha}[a_\alpha^2(\hat{p}_\alpha-\hat{p})^2+b_\alpha^2\hat{q}_\alpha^2], \qquad (A9)$$

the exact converse occurs [13–15]: for the harmonic oscillator this is immediately clear from the fact that the canonical transformation $\hat{p}\to-\hat{q}$ and $\hat{q}\to\hat{p}$ maps the two coupling models onto each other.

APPENDIX B: THE GENERAL GAUSSIAN INTEGRAL

Let us consider the following general quadratic action in Matsubara space,

$$\mathcal{S}=-\frac{\beta}{2}\sum_n[a_n^2p_np_{-n}+b_n^2q_nq_{-n}+2(\pi_n+\delta_n)p_nq_{-n}], \qquad (B1)$$

where $a_n=a_{-n}$, $b_n=b_{-n}$, and $\pi_n=\pi_{-n}$ are positive, while the antisymmetric part $\delta_n=-\delta_{-n}$ is such that $\delta_n\sim\nu_n$, the Matsubara frequency, for large n. The Gaussian average of any quantity $O(\{p_n,q_n\})$ is

$$\langle O\rangle=\frac{J}{\mathcal{Z}}\prod_n\int\frac{dp_ndq_n}{2\pi}O(\{p_n,q_n\})e^{\mathcal{S}}, \qquad (B2)$$

where $J=\Pi_{n=1}^{\infty}(\beta\nu_n)^2$ is the Jacobian of the Matsubara transformation, $\mathcal{Z}$ is the partition function, obtained for $O=1$, and the integrals are defined as ordinary integrals over the independent real and imaginary parts as

$$\prod_n\int dp_n\equiv\int dp_0\prod_{n=0}^{\infty}2\int dp_n^{\mathrm{R}}dp_n^{\mathrm{I}}, \qquad (B3)$$

where $p_{\pm n}\equiv p_n^{\mathrm{R}}\pm ip_n^{\mathrm{I}}$, and the analog for the coordinates.

The action can be easily worked out by means of simple transformations that preserve the measure:

(i) the coefficients of momenta and coordinates are made equal to $\omega_n=a_nb_n$ by

$$p_n=\sqrt{\frac{b_n}{a_n}}p_n^{(1)}, \qquad q_n=\sqrt{\frac{a_n}{b_n}}q_n^{(1)}, \qquad (B4)$$

(ii) the terms that multiply π_n are diagonalized to $p_n^{(2)}p_{-n}^{(2)}-q_n^{(2)}q_{-n}^{(2)}$ by the rotation

$$\begin{pmatrix}p_n^{(1)}\\ q_n^{(1)}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}p_n^{(2)}\\ q_n^{(2)}\end{pmatrix}, \qquad (B5)$$

(iii) so that a further balancing is in order,

$$p_n^{(2)}=\sqrt[4]{\frac{\omega_n-\pi_n}{\omega_n+\pi_n}}\tilde{p}_n, \qquad q_n^{(2)}\sqrt[4]{\frac{\omega_n+\pi_n}{\omega_n-\pi_n}}\tilde{q}_n, \qquad (B6)$$

eventually giving

$$\begin{aligned}\mathcal{S}&=-\frac{\beta}{2}\sum_n[\Omega_n(\tilde{p}_n\tilde{p}_{-n}+\tilde{q}_n\tilde{q}_{-n})+2\delta_n\tilde{p}_n\tilde{q}_{-n}]\\ &=-\frac{\beta}{2}\sum_n\left(\Omega_n\tilde{p}'_n\tilde{p}'_{-n}+\frac{\delta_n^2+\Omega_n^2}{\Omega_n}\tilde{q}_n\tilde{q}_{-n}\right),\end{aligned} \qquad (B7)$$

with $\Omega_n^2\equiv\omega_n^2-\pi_n^2=a_n^2b_n^2-\pi_n^2$ and $\tilde{p}'_n=\tilde{p}_n-(\delta_n/\Omega_n)\tilde{q}_n$; note that for Ω_n to be real one must require

$$\pi_n<\omega_n. \qquad (B8)$$

It is now straightforward to obtain the variances

$$\langle\tilde{p}_n\tilde{p}_{-n}\rangle=\langle\tilde{q}_n\tilde{q}_{-n}\rangle=\frac{1}{\beta}\frac{\Omega_n}{\delta_n^2+\Omega_n^2}$$

$$\langle\tilde{p}_n\tilde{q}_{-n}\rangle=\frac{1}{\beta}\frac{\delta_n}{\delta_n^2+\Omega_n^2}. \qquad (B9)$$

Integrating over $\tilde{p}'_n$ and $\tilde{q}_n$ one finds the partition function

$$\mathcal{Z}=\frac{1}{\beta\Omega_0}\prod_{n=1}^{\infty}\frac{\nu_n^2}{\delta_n^2+\Omega_n^2}. \qquad (B10)$$

It is easy to go backward through the transformations (iii), (ii), and (i) to obtain the explicit result for the original Matsubara variables of Eq. (B1),

$$\langle p_np_{-n}\rangle=\frac{1}{\beta}\frac{b_n^2}{\delta_n^2+a_n^2b_n^2-\pi_n^2}$$

$$\langle q_nq_{-n}\rangle=\frac{1}{\beta}\frac{a_n^2}{\delta_n^2+a_n^2b_n^2-\pi_n^2}$$

$$\langle p_nq_{-n}\rangle=\frac{1}{\beta}\frac{\delta_n-\pi_n}{\delta_n^2+a_n^2b_n^2-\pi_n^2}. \qquad (B11)$$

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